Chapter 9. Sequences and Series

Question-1

Write the first 5 terms of each of the following sequences:

(i) $a_n = (-1)^{n-1} 5^{n+1}$

(ii) $a_n = \frac{n(n^2+5)}{4}$

(iii) $a_n = -11n + 10$

(iv) $a_n = \frac{n+1}{n+2}$

(v) $a_n = \frac{1-(-1)^n}{3}$

(vi) $a_n = \frac{n^2}{3^n}$

Solution:

(i) $a_n = (-1)^{n-1} 5^{n+1}$

$a_1 = (-1)^0 5^2 = 5^2$;

$a_2 = (-1)^1 5^3 = -5^3$

$a_3 = (-1)^2 5^4 = 5^4$;

$a_4 = (-1)^3 5^5 = -5^5$

$a_5 = (-1)^4 5^6 = 5^6$

(ii) $a_n = \frac{n(n^2+5)}{4}$

$a_1 = \frac{1(1^2+5)}{4} = \frac{6}{4} = \frac{3}{2}$

$a_2 = \frac{2(4+5)}{4} = \frac{18}{4} = \frac{9}{2}$

$a_3 = \frac{3(9+5)}{4} = \frac{21}{2}$

$a_4 = \frac{4(16+5)}{4} = 21$

$a_5 = \frac{5(25+5)}{4} = \frac{75}{2}$

(iii) $a_n = -11n + 10$

$a_1 = -11 + 10 = -1$

$a_2 = -22 + 10 = -12$

$a_3 = -33 + 10 = -23$

$a_4 = -44 + 10 = -34$

$a_5 = -55 + 10 = -45$

(iv) $a_n = \frac{1+1}{1+2} = \frac{2}{3}$

$a_2 = \frac{2+1}{2+2} = \frac{3}{4}$

$a_3 = \frac{3+1}{3+2} = \frac{4}{5}$

$a_4 = \frac{4+1}{4+2} = \frac{5}{6}$

$a_5 = \frac{5+1}{5+2} = \frac{6}{7}$

(v) $a_n = \frac{1-(-1)^n}{3}$

$a_1 = \frac{1-(-1)^1}{3} = \frac{2}{3}$

$a_2 = \frac{1-(-1)^2}{3} = 0$

$a_3 = \frac{1-(-1)^3}{3} = \frac{2}{3}$

$a_4 = \frac{1-(-1)^4}{3} = 0$

$a_5 = \frac{1-(-1)^5}{3} = \frac{2}{3}$

(vi) $a_n = \frac{n^2}{3^n}$

$a_1 = \frac{1^2}{3^1}$; $a_2 = \frac{2^2}{3^2}$; $a_3 = \frac{3^2}{3^3}$; $a_4 = \frac{4^2}{3^4}$; $a_5 = \frac{5^2}{3^5}$

Question-2

Find the first terms of the following sequences whose n^{th} term is

(i) $a_n = 2+\frac{1}{n}$; a_5, a_7

(ii) $a_n = \cos\left[\frac{n\pi}{2}\right]$; a_4, a_5

(iii) $a_n = \frac{(n+1)^2}{n}$; a_7, a_{10}

(iv) $a_n = (-1)^{n-1}\, 2^{n+1}$; a_5, a_8

Solution:

(i) $a_n = 2 + \frac{1}{n}$

$a_5 = 2 + \frac{1}{5}$

$= \frac{11}{5}$;

$a_7 = 2 + \frac{1}{7}$

$= \frac{15}{7}$

(ii) $a_n = \cos\left[\frac{n\pi}{2}\right]$;

$a_4 = \cos\left(\frac{4\pi}{2}\right) = \cos 2\pi = 1$

$a_5 = \cos\left[\frac{5\pi}{2}\right] = \cos\left[2\pi + \frac{\pi}{2}\right] = \cos\frac{\pi}{2} = 0$

(iii) $a_n = \frac{(n+1)^2}{n}$

$a_7 = \frac{(7+1)^2}{7}$

$= \frac{64}{7}$;

$a_{10} = \frac{(10+1)^2}{10}$

$= \frac{121}{10}$

(iv) $a_n = (-1)^{n-1}\, 2^{n+1}$

$a_5 = (-1)^4\, 2^{5+1}$

$= 2^6$

$= 64$;

$a_8 = (-1)^7\, 2^{8+1}$

$= -\,2^9$

$= -512$

Question-3

Find the first 6 terms of the sequence whose general term is $a_n = \{n^2 - 1$ if n is odd $\frac{n^2+1}{2}$ if n is even$\}$

Solution:

$a_1 = 1^2 - 1 = 0$

$a_2 = \frac{2^2+1}{2} = \frac{5}{2}$

$a_3 = 3^2 - 1 = 8$

$a_4 = \frac{4^2+1}{2} = \frac{17}{2}$

$a_5 = 5^2 - 1 = 24$

$a_6 = \frac{6^2+1}{2} = \frac{37}{2}$

Question-4

Write the first five terms of the sequence given by

(i) $a_1 = a_2 = 2, a_n = a_{n-1} - 1, n>2$

(ii) $a_1 = 1, a_2 = 2, a_n = a_{n-1} + a_{n-2}, n>2$

(iii) $a_1 = 1, a_n = na_{n-1}, n \geq 2$

(iv) $a_1 = a_2 = 1, a_n = 2a_{n-1} + 3a_{n-2}, n>2$

Solution:

(i) Put $n = 3 \Rightarrow a_3 = a_2 - 1 = 2-1 = 1$

$n = 4 \Rightarrow a_4 = a_3 - 1 = 1-1 = 0$

$n = 5 \Rightarrow a_5 = a_4 - 1 = 0-1 = -1$

(ii) Put $n = 3 \Rightarrow a_3 = a_2 + a_1 = 2 + 1 = 3$

$n = 4 \Rightarrow a_4 = a_3 + a_2 = 3 + 2 = 5$

$n = 5 \Rightarrow a_5 = a_4 + a_3 = 5 + 3 = 8$

(iii) Put $n = 2 \Rightarrow a_2 = 2\,a_1 = 2.1 = 2$

$n = 3 \Rightarrow a_3 = 3\,a_2 = 3.2 = 6$

$n = 4 \Rightarrow a_4 = 4\,a_3 = 4.6 = 24$

$n = 5 \Rightarrow a_5 = 5\,a_4 = 5.24 = 120$

(iv) Put $n = 3 \Rightarrow a_3 = 2a_2 + 3a_1 = 2(1) + 3(1) = 5$

$n = 4 \Rightarrow a_4 = 2a_3 + 3a_2 = 2(5) + 3(1) = 13$

$n = 5 \Rightarrow a_5 = 2a_4 + 3a_3 = 2(13) + 3(5) = 41$

Question-5

Find the n^{th} partial sum of the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$

Solution:

$\sum_{n=1}^{\infty} \frac{1}{3^n}$

$S_n = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\ldots\ldots\ldots \frac{1}{3^n}$

$S_{n+1} = \frac{1}{3} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots + \frac{1}{3^n} + \frac{1}{3^{n+1}}$

$S_{n+1} = S_n + \frac{1}{3^{n+1}}$

$S_{n+1} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\ldots\ldots\ldots \frac{1}{3^n} + \frac{1}{3^{n+1}}$

$= \frac{1}{3}\left[1 + \frac{1}{3} + \frac{1}{3^2} + \ldots + \frac{1}{3^n}\right] = \frac{1}{3}\left[1 + s_n\right]$

$s_n + \frac{1}{3^{n+1}} = \frac{1}{3} + \frac{1}{3} s_n$

$3 s_n + \frac{1}{3^n} = 1 + s_n$

$2s_n = 1 - \frac{1}{3^n}$

$s_n = \frac{1}{2}\left[1 - \frac{1}{3^n}\right]$

Question-6

Find the sum of first n terms of the series $\sum_{n=1}^{\infty} 5^n$

Solution:

$\sum_{n=1}^{\infty} 5^n = 5 + 5^2 + 5^3 + \ldots\ldots\ldots + 5^n + \ldots\ldots\ldots$

$s_n = 5 + 5^2 + 5^3 + \ldots\ldots\ldots + 5^n$

$s_{n+1} = 5 + 5^2 + 5^3 + \ldots\ldots\ldots + 5^n + 5^{n+1}$

$= s_n + 5^{n+1}$

Also $s_{n+1} = 5 + 5^2 + 5^3 + \ldots\ldots\ldots + 5^n + 5^{n+1}$

$= 5[1 + 5 + 5^2 + \ldots\ldots\ldots + 5^n]$

$= 5[1 + s_n]$

$s_n + 5^{n+1} = 5 + 5 s_n$

$4s_n = 5^{n+1} - 5$

$\therefore s_n = \frac{5(5^n - 1)}{4}$

Question-7

Find the sum of 101th term to 200th term of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$

Solution:

$\sum_{n=1}^{\infty} \frac{1}{2^n}$

To find $S_{200} - S_{100}$

To find S_{200}: $S_{200} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\dots\dots + \frac{1}{2^{200}}$

$S_{201} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\dots\dots + \frac{1}{2^{200}} + \frac{1}{2^{201}}$

$= S_{200} + \frac{1}{2^{201}}$

also, $S_{201} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\dots\dots + \frac{1}{2^{200}} + \frac{1}{2^{201}}$

$= \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \dots\dots \frac{1}{2^{200}}\right]$

$S_{200} + \frac{1}{2^{201}} = \frac{1}{2}[1 + S_{200}]$

$2S_{200} + \frac{1}{2^{200}} = 1 + S_{200}$

$S_{200} = 1 - \frac{1}{2^{200}}$

Similarly $S_{100} = 1 - \frac{1}{2^{100}}$

Hence $S_{200} - S_{100} = \left[1 - \frac{1}{2^{200}}\right] - \left[1 - \frac{1}{2^{100}}\right]$

$= \frac{1}{2^{100}} - \frac{1}{2^{200}}$

Question-8

Find five arithmetic means between 1 and 19.

Solution:

Let 1, x_1, x_2, x_3, x_4, x_5, 19 be in A.P.

Let d be the common difference

19 = 1 + (n-1)d

19 = 1 + 6d

$\therefore$ d = 3

$\therefore x_1 = 1 + 3 = 4$

$x_2 = 4 + 3 = 7$

$x_3 = 7 + 3 = 10$

$x_4 = 10 + 3 = 13$

$x_5 = 13 + 3 = 16$

The arithmetic means are 4, 7, 10, 13, 16.

Question-9

Find six arithmetic mean between 3 and 17.

Solution:

Let 3, x_1, x_2, x_3, x_4, x_5, x_6, 17 be in A.P

Then 17 = 3 + (n-1)d

17 = 3 + 7d

14 = 7d

d = 2

$x_1 = 3 + 2 = 5$

$x_2 = 5 + 2 = 7$

$x_3 = 7 + 2 = 9$

$x_4 = 9 + 2 = 11$

$x_5 = 11 + 2 = 13$

$x_6 = 13 + 2 = 15$

The arithmetic means are 5, 7, 9, 11, 13, 15.

Question-10

Find the single A.M. between

(i) 7 and 13

(ii) 5 and −3

(iii) (p + q) and (p - q)

Solution:

(i) A.M. between 7 and 13 = $\frac{7+13}{2}$ = 10

(ii) A.M. between 5 and −3 = $\frac{5-3}{2}$ = 1

(iii) A.M. between (p + q) and (p − q) = $\frac{p+q+p-q}{2}$ = p

Question-11

If b is the G.M. of a and c and x is the A.M of a and b and y is the A.M of b and c, prove that $\frac{a}{x}+\frac{c}{y} = 2$.

Solution:

b = G.M. of a and c $\Rightarrow \sqrt{ac} = b$ ....(1)

x = A.M. of a and b $\Rightarrow x = \frac{a+b}{2}$ ....(2)

y = A.M. between b and c $\Rightarrow y = \frac{b+c}{2}$ ....(3)

To prove that $\frac{a}{x}+\frac{b}{y} = 2$

From (1) $b^2 = ac \Rightarrow c = \frac{b^2}{a}$

$$\frac{a}{x}+\frac{c}{y} = \frac{2a}{a+b} + \frac{2c}{b+c}$$

$$= \frac{2a}{a+b} + \frac{\frac{2b^2}{a}}{b+\frac{b^2}{a}}$$

$$= \frac{2a}{a+b} + \frac{2bb}{b(a+b)}$$

$$= \frac{2a}{a+b} + \frac{2b}{a+b}$$

$$= \frac{2(a+b)}{a+b}$$

$$= 2$$

Question-12

The first and second terms of H.P are $\frac{1}{3}$ and $\frac{1}{5}$ respectively, find the 9^{th} term.

Solution:

Let the H.P are $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d} + \ldots\ldots\ldots$

$$\frac{1}{a} = \frac{1}{3} \Rightarrow a = 3$$

$$\frac{1}{a+d} = \frac{1}{5} \Rightarrow 5 = a + d;\ d = 2$$

9^{th} term $= \frac{1}{a+8d} = \frac{1}{3+16} = \frac{1}{19}$

Question-13

If a, b, c are in H.P., prove that $\frac{b+a}{b-a} + \frac{b+c}{b-c} = 2$.

Solution:

If a, b, c are in H.P then $b = \frac{2ac}{a+c}$

$\frac{b}{a} = \frac{2c}{a+c}$

$\Rightarrow \frac{b+a}{b-a} = \frac{2c+a+c}{2c-a-c} = \frac{3c+a}{c-a}$ ....(1)

Also $\frac{b}{c} = \frac{2a}{a+c}$

$\frac{b+c}{b-c} = \frac{2a+a+c}{2a-a-c} = \frac{3a+c}{a-c}$ ....(2)

Adding (1) and (2)

$\therefore \frac{b+a}{b-a} + \frac{b+c}{b-c} = \frac{3c+a}{c-a} + \frac{3a+c}{a-c}$

$= \frac{3c+a}{c-a} - \frac{3a+c}{c-a}$

$= \frac{3c+a-3a-c}{c-a}$

$= \frac{2c-2a}{c-a}$

$= 2$

Hence $\frac{b+a}{b-a} + \frac{b+c}{b-c} = 2$.

Question-14

The difference between two positive numbers is 18, and 4 times their G.M. is equal to 5 times their H.M. find the numbers.

Solution:

Let the two numbers be a and b

$b - a = 18$

$4\sqrt{ab} = 5\left[\frac{2ab}{a+b}\right]$

$2\sqrt{ab} = \frac{5ab}{a+b}$

$2(a+b) = 5\sqrt{ab}$

$4(a+b)^2 = 25ab$

$4(a^2 + b^2 + 2ab) = 25ab$

$4a^2 + 4b^2 - 17ab = 0$

$4a^2 + 4(18+a)^2 - 17a(18+a) = 0$

$4a^2 + 4(324 + 36a + a^2) - 306a - 17a^2 = 0$

$4a^2 + 1296 + 144a + 4a^2 - 306a - 17a^2 = 0$

$-9a^2 - 162a + 1296 = 0$

$a^2 + 18a - 144 = 0$

$(a+24)(a-6) = 0$

$a = -24$ (or) 6

If a = 6 then b is 24.

Therefore the numbers are 6 and 24.

Question-15

If the A.M. between two numbers is 1, prove that their H.M. is the square of their G.M.

Solution:

A.M. between two numbers a and b is 1.

$\frac{a+b}{2} = 1$

$\Rightarrow a + b = 2$

$HM = (GM)^2$

$HM = \frac{2ab}{a+b} = \frac{2ab}{2} = ab$

$GM = \sqrt{ab}$

$\therefore (GM)^2 = ab$

Hence $HM = GM^2$

Question-16

If a, b, c are in A.P and a, mb, c are in G.P then prove that a, m^2b, c are in H.P.

Solution:

Given

a, b, c are in A.P.

$\Rightarrow b = \frac{a+c}{2}$ ....(1)

a, mb, c are in G.P.

$\Rightarrow mb = \sqrt{ac}$ ....(2)

To prove

a,m^2b,c are in H.P.

i.e., $m^2b = \frac{2ac}{a+c}$

Proof

$R.H.S = \frac{2ac}{a+c} = \frac{2m^2b^2}{2b}$ from (2) and (1)

$= m^2 b = LHS$

Question-17

If the p^{th} and q^{th} terms of a H.P. are q and p respectively, show that $(pq)^{th}$ term is 1.

Solution:

Given

p^{th} and q^{th} terms of a H.P. are q and p.

Therefore $\frac{1}{a+(p-1)d} = q$ ....(1)

and $\frac{1}{a+q-1d} = p$ ....(2)

To prove

pq^{th} term, ie., $\frac{1}{a+(pq-1)d} = 1$

Proof

From (1) $a + pd - d = \frac{1}{q}$

From (2) $a + qd - d = \frac{1}{p}$

Subtracting $(p - q)d = \frac{1}{q} - \frac{1}{p} = \frac{p-q}{pq}$

$$\therefore d = \frac{1}{pq}$$

$$\therefore a + p\frac{1}{pq} - \frac{1}{pq} = \frac{1}{q}$$

$$a = \frac{1}{pq}$$

$$\therefore a + (pq - 1)d = \frac{1}{pq} + (pq - 1)\frac{1}{pq}$$

$$= \frac{1 + pq - 1}{pq}$$

$$= \frac{pq}{pq} = 1$$

$$\therefore \frac{1}{a + (pq-1)d} = 1$$

⇒ pq^{th} term is 1.

Question-18

Three number form a H.P. the sum of the numbers is 11 and the sum of the reciprocals is one. Find the numbers.

Solution:

Let $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$ be in H.P.

Their sum is $\frac{1}{a-d} + \frac{1}{a} + \frac{1}{a+d} = 11$

The sum of their reciprocal is $a - d + a + a + d = 1$

$3a = 1$

⇒ $a = \frac{1}{3}$

$$\therefore \frac{1}{\frac{1}{3}-d} + \frac{1}{\frac{1}{3}} + \frac{1}{\frac{1}{3}+d} = 11$$

$$\frac{3}{1-3d} + 3 + \frac{3}{1+3d} = 11$$

$$\frac{3}{1-3d} + \frac{3}{1+3d} = 8$$

$$\frac{3(1+3d+1-3d)}{1-9d^2} = 8$$

$$\frac{6}{1-9d^2} = 8$$

$$6 = 8 - 72d^2$$

$$72d^2 = 2$$

$$\Rightarrow d^2 = \frac{1}{36}$$

$$\Rightarrow d = \frac{1}{6}$$

The numbers are $\frac{1}{\frac{1}{3}-\frac{1}{6}} + \frac{1}{\frac{1}{3}} + \frac{1}{\frac{1}{3}+\frac{1}{6}} = \frac{1}{\frac{1}{6}}, \frac{1}{\frac{1}{3}}, \frac{1}{\frac{1}{2}}$

The numbers are 6, 3, 2.

Question-19

Write the first four terms in the expansions of the following:

(i) $\frac{1}{(2+x)^4}$ where $|x| > 2$

(ii) $\frac{1}{\sqrt[3]{6-3x}}$ where $|x| < 2$

Solution:

(i) $\frac{1}{(2+x)^4} = \frac{1}{16\left(1+\frac{x}{2}\right)^4} = \frac{1}{16}\left[1+\frac{x}{2}\right]^{-4}$

$= \frac{1}{16}\left[1 - 4\left(\frac{x}{2}\right) + \frac{4.5}{1.2}\left(\frac{x}{2}\right)^2 - \frac{4.5.6}{1.2.3}\left(\frac{x}{2}\right)^3 + \ldots\right]$

$= \frac{1}{16}\left[1 - 2x + \frac{5}{2}x^2 - \frac{5}{2}x^3 + \ldots.\right]$

(ii) $\frac{1}{\sqrt[3]{6-3x}} = \frac{1}{(6-7x)^{\frac{1}{3}}} = \frac{1}{6^{\frac{1}{3}}}\left[1 - \frac{x}{2}\right]^{\frac{1}{3}}$

$= \frac{1}{6^{\frac{1}{3}}}\left[1 + \frac{1}{3}\left(\frac{x}{2}\right) + \frac{\left(\frac{1}{3}\right)\left(\frac{4}{3}\right)}{1.2}\left(\frac{x}{2}\right)^2 + \frac{\left[\frac{1}{3}\right]\left[\frac{4}{3}\right]\left[\frac{7}{3}\right]}{1.2.3}\left[\frac{x}{2}\right]^3\right]$

$= \frac{1}{6^{\frac{1}{3}}}\left[1 + \frac{x}{6} + \frac{x^2}{18} + \frac{7}{326}x^3 + \ldots.\right]$

Question-20

Evaluate the following:

(i) $\sqrt[3]{1003}$ correct to 4 places of decimals.

(ii) $\frac{1}{\sqrt[3]{128}}$ correct to 4 places of decimals.

(iii) $\sqrt[3]{1003}$ - $\sqrt[3]{997}$ correct to 3 place of decimals.

Solution:

(i) $\sqrt[3]{1003} = (1003)^{\frac{1}{3}}$

$$= (1000 + 3)^{\frac{1}{3}}$$

$$= (1000)^{1/3}\left[1 + \frac{3}{1000}\right]^{\frac{1}{3}}$$

$$= 10\,[1+0.003]^{\frac{1}{3}}$$

$$= 10\left[1+\frac{1}{3}(0.003)+\frac{\left[\frac{1}{3}\right]\left[\frac{-2}{3}\right]}{1.2}(0.003)^2+....\right]$$

$$= 10\,[1+0.001-0.000001+....1]$$

$$= 10.00999$$

(ii) $\frac{1}{\sqrt[3]{128}} = \frac{1}{(128)^{\frac{1}{3}}} = \frac{1}{(125+3)^{\frac{1}{3}}} = \frac{1}{5\left[1+\frac{3}{125}\right]^{\frac{1}{3}}} = \frac{1}{5}\left[1+\frac{3}{125}\right]^{\frac{1}{3}}$

$$= \frac{1}{5}\,(1 + 0.024)^{-\frac{1}{3}} = \frac{1}{5}\left[1-\frac{1}{3}(0.024)+\frac{\frac{1}{3}\left[\frac{4}{3}\right]}{1.2}(0.024)^2.....\right]$$

$$= \frac{1}{5}\,[1-0.008+0.000128] = 0.1984256$$

(iii) $\sqrt[3]{1003} - \sqrt[3]{997} = (1000 + 3)^{1/3} - (1000 - 3)^{1/3}$

$$= 10\left[1+\frac{3}{1000}\right]^{\frac{1}{3}} - 10\left[1-\frac{3}{1000}\right]^{\frac{1}{3}}$$

$$= 10\,[1 + 0.003]^{\frac{1}{3}} - 10[1 - 0.003]^{\frac{1}{3}}$$

$$= 10\left[1+\frac{1}{3}(0.003)+\frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)}{1.2}(0.003)^2+....\right]$$

$$-10\left[1-\frac{1}{3}(0.003)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{1.2}(0.003)^2...\right]$$

$$= 10\left[1+\frac{0.003}{3}-\frac{(0.003)^2}{9}-1+\frac{0.003}{3}+\frac{(0.003)^2}{9}....\right]$$

$$= 10[0.002] = 0.02$$

Question-21

If x is so small show that

(i) $\frac{\sqrt{1-x}}{1+x} = 1 - x + \frac{x^2}{2}$(app)

(ii) $\frac{1}{(1+x)^2\sqrt{1+4x}}$ =1 - 4x(app.)

Solution:

(i) $\frac{\sqrt{1-x}}{1+x} = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$

$= \left[1 - \frac{1}{2}x + \frac{\left[\frac{1}{2}\right]\left[\frac{-1}{2}\right]}{1.2}x^2 + ...\right]\left[1 - \frac{1}{2}x + \frac{\frac{1}{2}\frac{3}{2}}{1.2}x^2 + ...\right]$

$= \left[1 - \frac{x}{2} - \frac{x^2}{8}\right]\left[1 - \frac{x}{2} + \frac{3x^2}{8}\right]$

$= 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x}{2} + \frac{x^2}{4} + \frac{3x^2}{8} + ...$

$= 1 - x + \frac{x^2}{2}$ (app.)

(ii) $\frac{1}{(1+x)^2\sqrt{1+4x}} = (1+x)^{-2}(1+4x)^{-1/2}$

$= \left[1 - 2x + \frac{2.3}{1.2}x^2 + ...\right]\left[1 - \frac{1}{2}(4x) + ...\right]$

$= (1 - 2x + ...)(1 - 2x + ...)$

$= 1 - 2x - 2x + 4x^2 + ... = 1 - 4x$(app.)

Question-22

If x is so large prove that $\sqrt{x^2+25} - \sqrt{x^2+9} = \frac{8}{x}$ nearly.

Solution:

$\sqrt{x^2+25} - \sqrt{x^2+9} = x\left[1 + \frac{25}{x^2}\right]^{\frac{1}{2}} - x\left[1 + \frac{9}{x^2}\right]^{\frac{1}{2}}$

$= x\left[1 + \frac{1}{2}\left[\frac{25}{x^2}\right] + \frac{\frac{1}{2}\left[\frac{-1}{2}\right]}{1.2}\left[\frac{25}{x^2}\right]^2 + ...\right] - x\left[1 + \frac{1}{2}\left[\frac{9}{x^2}\right] + \frac{\frac{1}{2}\left[\frac{-1}{2}\right]}{1.2}\left[\frac{9}{x^2}\right]^2 + ...\right]$

$= x + \frac{25}{2x} - \frac{625}{8x^3} + ... - x - \frac{9}{2x} + \frac{81}{8x^3} + ...$

$= \frac{16}{2x} + \frac{81}{8x^3} + ...$

$= \frac{16}{2x} = \frac{8}{x}$ approximately

Question-23

If c is small compared to l, show that $\left[\frac{l}{l+c}\right]^{\frac{1}{2}} + \left[\frac{l}{l-c}\right]^{\frac{1}{2}} = 2 + \frac{3c^2}{4l^2}$ (app)

Solution:

$$\left[\frac{l}{l+c}\right]^{\frac{1}{2}} + \left[\frac{l}{l-c}\right]^{\frac{1}{2}} = \frac{1}{\left[1+\frac{c}{l}\right]^{\frac{1}{2}}} + \frac{1}{\left[1-\frac{c}{l}\right]^{\frac{1}{2}}}$$

$$= \left[1+\frac{c}{l}\right]^{\frac{1}{2}} + \left[1-\frac{c}{l}\right]^{\frac{-1}{2}}$$

Since c is small in comparison with l then $\left|\frac{c}{l}\right| < 1, \therefore$ binomial expansion is valid.

$$= 1+\left[\frac{-1}{2}\right]\left[\frac{c}{l}\right]+\frac{\left[\frac{-1}{2}\right]\left[\frac{-1}{2}-1\right]}{1.2}\left[\frac{-c}{l}\right]^2+\ldots\ldots\ldots$$

$$+1+\left[\frac{-1}{2}\right]\left[\frac{-c}{l}\right]+\frac{\left[\frac{-1}{2}\right]\left[\frac{-1}{2}-1\right]}{1.2}\left[\frac{-c}{l}\right]^2+\ldots\ldots\ldots$$

$$= 1 - \frac{c}{2l} + \frac{3c^2}{8l^2} + \ldots\ldots + 1 + \frac{c}{2l} + \frac{3c^2}{8l^2} + \ldots.$$

$= 2 + \frac{3c^2}{4l^2}$ approximately.

Question-24

Find the 5th term in the expansion of $(1 - 2x^3)^{11/2}$.

Solution:

$$(1 - 2x^3)^{11/2} = 1 + \left[\frac{11}{2}\right](-2x^3) + \frac{\left[\frac{11}{2}\right]\left[\frac{2}{9}\right]}{1.2}(-2x^3)^2 + \frac{\left[\frac{11}{2}\right]\left[\frac{2}{9}\right]\left[\frac{7}{2}\right]}{1.2.3}(-2x^3)^3 + \frac{\left[\frac{11}{2}\right]\left[\frac{2}{9}\right]\left[\frac{7}{2}\right]\left[\frac{5}{2}\right]}{1.2.3.4}(-2x^3)^4$$

5th term is $\frac{\left[\frac{11}{2}\right]\left[\frac{2}{9}\right]\left[\frac{7}{2}\right]\left[\frac{5}{2}\right]}{1.2.3.4}(-2x^3)^4 = \frac{1155}{8}x^{12}$

Question-25

Find the (r + 1)th term in the expansion of $(1 - x)^{-4}$.

Solution:

T_{r+1} in $(1 - x)^{-4}$

$(1-x)^{-4} = \frac{1}{6}[1.2.3 + 2.3.4.x + \ldots\ldots\ldots\ldots(r + 1)(r + 2)(r + 3)x^r + \ldots\ldots\ldots]$

$\therefore T_{r+1} = \frac{(r+1)(r+2)(r+3)}{6}x^r$

Question-26

Show that $x^n = 1 + n\left[1-\frac{1}{x}\right] + \frac{n(n+1)}{1.2}\left[1-\frac{1}{x}\right]^2 + \ldots..$

Solution:

R.H.S $= 1 + n\left[1-\frac{1}{x}\right] + \frac{n(n+1)}{1.2}\left[1-\frac{1}{x}\right]^2 + \ldots..$

Put $y = 1-\frac{1}{x}$

$= 1 + ny + \frac{n(n+1)}{1.2}y^2 + \ldots.$

$= (1-y)^{-n}$

$= \left[1-\left[1-\frac{1}{x}\right]\right]^{-n}$

$= \left[\frac{1}{x}\right]^{-n}$

$= x^n$

= L.H.S

Question-27

Find the sum to infinity of the series

(i) $1 + \frac{9}{8} + \frac{9.15}{8.16} + \frac{9.15.21}{8.16.24} + \ldots$

(ii) $1 - \frac{1}{5} + \frac{1.4}{5.10} - \frac{1.4.7}{5.10.15} + \ldots.$

(iii) $\frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \ldots$

Solution:

(i) Let $S = 1 + \frac{9}{8} + \frac{9.15}{8.16} + \frac{9.15.21}{8.16.24} + \ldots$

$= 1 + \frac{9}{6}\left(\frac{6}{8}\right) + \frac{\left(\frac{9}{6}\right)\left(\frac{5}{6}\right)}{1.2}\left(\frac{6}{8}\right)^2 + \ldots\ldots..$

$= \left[1-\frac{6}{8}\right]^{-\frac{9}{6}}$

$\left[1-\frac{3}{4}\right]^{-\frac{3}{2}} = \left[\frac{1}{4}\right]^{-\frac{3}{2}} = 4^{\frac{3}{2}} = 4^1\, 4^{\frac{1}{2}} = 4\sqrt{4} = 4(2) = 8$

(ii) Let $S = 1 - \frac{1}{5} + \frac{1.4}{5.10} - \frac{1.4.7}{5.10.15} + ...$

$= 1 - \left(\frac{1}{3}\right)\left(\frac{3}{5}\right) + \frac{\left[\frac{1}{3}\right]\left[\frac{4}{3}\right]}{1.2}\left[\frac{3}{5}\right]^2 + ...$

$= \left[1 + \frac{3}{5}\right]^{-\frac{1}{3}}$

$= \left[\frac{8}{5}\right]^{-\frac{1}{3}}$

$= \left[\frac{5}{8}\right]^{\frac{1}{3}}$

(iii) Let $S = \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + ...$

$S + 1 = 1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + ...$

$= 1 + \left(\frac{3}{2}\right)\left(\frac{2}{4}\right) + \frac{\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{1.2}\left(\frac{2}{4}\right)^2 + ...$

$= \left(1 - \frac{2}{4}\right)^{-3/2} = \left(\frac{1}{2}\right)^{-3/2} = 2^{3/2}$

$S + 1 = 2^{3/2}$

Therefore $S = 2^{3/2} - 1$

Question-28

Show that the coefficient of x^n in the infinite series $1 + \frac{b + ax}{1!} + \frac{(b + ax)^2}{2!} + \frac{(b + ax)^3}{3!} + ...$ is $\frac{e^b a^n}{n!}$.

(ii) Show that $\left(1 + \frac{1}{2!} + \frac{1}{4!} + ...\right)^2 = 1 + \left(1 + \frac{1}{3!} + \frac{1}{5!} + ...\right)^2$.

(iii) Show that $2\left\{1 + \frac{(\log n)^2}{2!} + \frac{(\log n)^4}{4!} + ...\right\} = n + c.$

Solution:

(i) $1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + ... = e^y = e^{b + ax} = e^b \cdot e^{ax} = e^b\left(1 + \frac{ax}{1!} + \frac{(ax)^2}{2!} + ...\right)$

Coefficient of $x^n = e^b \cdot \left(\frac{a^2}{n!}\right)$

(ii) L.H.S = $\left(1+\frac{1}{2!}+\frac{1}{4!}+\ldots\ldots\right)^2 = \left(\frac{e+e^{-1}}{2}\right)^2$

$= 1+\frac{e^2+e^{-2}-2}{4}$

$= \frac{4+e^2+e^{-2}-2}{4}$

$= \frac{e^2+e^{-2}+2}{4}$

= R.H.S

(iii) L.H.S = $2\left\{1+\frac{(\log n)^2}{2!}+\frac{(\log n)^4}{4!}+\ldots\right\}$

Put log n = y

$2\left\{1+\frac{(\log n)^2}{2!}+\frac{(\log n)^4}{4!}+\ldots\right\} = 2\left(\frac{e^y+e^{-y}}{2}\right)$

$= e^y + e^{-y}$

$= e^{\log n} + e^{-\log n}$

$= e^{\log n} + e^{\log 1/n}$

$= n + \frac{1}{n}$

Question-29

Show that log a – log b = $\frac{a-b}{a}+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\ldots\ldots\ldots\ldots$

Solution:

R.H.S = $\frac{a-b}{a}+\frac{1}{2}\left(\frac{a-b}{a}\right)^2+\frac{1}{3}\left(\frac{a-b}{a}\right)^3+\ldots\ldots\ldots\ldots$

Put y = $\frac{a-b}{a}$

$y+\frac{y^2}{2}+\frac{y^3}{3}+\ldots\ldots\ldots\ldots = -\log(1-y)$

$= -\log\left(1-\frac{a-b}{a}\right)$

$= -\log\left(\frac{b}{a}\right)$

$= \log\left(\frac{a}{b}\right)$

= log a – log b

= L.H.S

Question-30

Prove that $\log \frac{n+1}{n-1} = \frac{2n}{n^2+1} + \frac{1}{3}\left(\frac{2n}{n^2+1}\right)^3 + \ldots\ldots\ldots\ldots$

Solution:

R.H.S = $\frac{2n}{n^2+1} + \frac{1}{3}\left(\frac{2n}{n^2+1}\right)^3 + \ldots\ldots\ldots\ldots$

Put $\frac{2n}{n^2+1} = y$

$y + \frac{y^3}{3} + \frac{y^5}{5} \ldots\ldots\ldots\ldots = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right)$

$= \frac{1}{2}\log\left(\frac{1+\frac{2n}{n^2+1}}{1-\frac{2n}{n^2+1}}\right)$

$= \frac{1}{2}\log\left(\frac{n^2+1+2n}{n^2+1-2n}\right)$

$= \frac{1}{2}\log\left(\frac{n+1}{n-1}\right)^2$

$= \log\left(\frac{n+1}{n-1}\right)$

= L.H.S

Question-31

Find the sum to infinity the series $\frac{1}{x-1} + \frac{1}{3}\frac{1}{(x-1)^3} + \frac{1}{5}\frac{1}{(x-1)^5} + \ldots\ldots\ldots\ldots\ldots$

Solution:

$\frac{1}{x-1} + \frac{1}{3}\frac{1}{(x-1)^3} + \frac{1}{5}\frac{1}{(x-1)^5} + \ldots\ldots\ldots\ldots\ldots$

Put $y = \frac{1}{x-1}$

$\frac{1}{y} + \frac{1}{3}\frac{1}{y^3} + \frac{1}{5}\frac{1}{y^5} + \ldots\ldots\ldots\ldots = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right)$

$= \frac{1}{2}\log\left(\frac{1+\frac{1}{1-x}}{1-\frac{1}{1-x}}\right)$

$= \frac{1}{2}\log\left(\frac{1-x+1}{1-x-1}\right)$

$= \frac{1}{2}\log\left(\frac{2-x}{-x}\right)$

$= \frac{1}{2}\log\left(\frac{x+2}{x}\right)$

Question-32

If x is so small show that

(i) $\frac{\sqrt{1-x}}{1+x} = 1 - x + \frac{x^2}{2}$ (app)

(ii) $\frac{1}{(1+x)^2\sqrt{1+4x}} = 1 - 4x$(app.)

Solution:

(i) $\frac{\sqrt{1-x}}{1+x} = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$

$$= \left[1 - \frac{1}{2}x + \frac{\left[\frac{1}{2}\right]\left[\frac{-1}{2}\right]}{1.2}x^2 + \dots\right]\left[1 - \frac{1}{2}x + \frac{\frac{1}{2}\frac{3}{2}}{1.2}x^2 + \dots\right]$$

$$= \left[1 - \frac{x}{2} - \frac{x^2}{8}\right]\left[1 - \frac{x}{2} + \frac{3x^2}{8}\right]$$

$$= 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x}{2} + \frac{x^2}{4} + \frac{3x^2}{8} + \dots$$

$$= 1 - x + \frac{x^2}{2} \text{ (app.)}$$

(ii) $\frac{1}{(1+x)^2\sqrt{1+4x}} = (1+x)^{-2}(1+4x)^{-1/2}$

$$= \left[1 - 2x + \frac{2.3}{1.2}x^2 + \dots\right]\left[1 - \frac{1}{2}(4x) + \dots\right]$$

$$= (1 - 2x + \dots\dots)(1 - 2x + \dots\dots)$$

$$= 1 - 2x - 2x + 4x^2 + \dots = 1 - 4x \text{(app.)}$$

Question-33

If x is so large prove that $\sqrt{x^2+25} - \sqrt{x^2+9} = \frac{8}{x}$ nearly.

Solution:

$$\sqrt{x^2+25} - \sqrt{x^2+9} = x\left[1 + \frac{25}{x^2}\right]^{\frac{1}{2}} - x\left[1 + \frac{9}{x^2}\right]^{\frac{1}{2}}$$

$$= x\left[1 + \frac{1}{2}\left[\frac{25}{x^2}\right] + \frac{\frac{1}{2}\left[\frac{-1}{2}\right]}{1.2}\left[\frac{25}{x^2}\right]^2 + \dots\right] - x\left[1 + \frac{1}{2}\left[\frac{9}{x^2}\right] + \frac{\frac{1}{2}\left[\frac{-1}{2}\right]}{1.2}\left[\frac{9}{x^2}\right]^2 + \dots\right]$$

$$= x + \frac{25}{2x} - \frac{625}{8x^3} + \dots - x - \frac{9}{2x} + \frac{81}{8x^3} + \dots$$

$$= \frac{16}{2x} + \frac{81}{8x^3} + \dots$$

$$= \frac{16}{2x} = \frac{8}{x} \text{ approximately}$$

Question-34

If c is small compared to l, show that $\left[\frac{1}{l+c}\right]^{\frac{1}{2}} + \left[\frac{1}{l-c}\right]^{\frac{1}{2}} = 2 + \frac{3c^2}{4l^2}$ (app)

Solution:

$$\left[\frac{l}{l+c}\right]^{\frac{1}{2}} + \left[\frac{1}{1-c}\right]^{\frac{1}{2}} = \frac{1}{\left[1+\frac{c}{l}\right]^{\frac{1}{2}}} + \frac{1}{\left[1-\frac{c}{l}\right]^{\frac{1}{2}}}$$

$$= \left[1+\frac{c}{l}\right]^{\frac{1}{2}} + \left[1-\frac{c}{l}\right]^{\frac{-1}{2}}$$

Since c is small in comparison with l then $|\frac{c}{l}| < 1$, ∴ binomial expansion is valid.

$$= 1+\left[\frac{-1}{2}\right]\left[\frac{c}{l}\right]+\frac{\left[\frac{-1}{2}\right]\left[\frac{-1}{2}-1\right]}{1.2}\left[\frac{-c}{l}\right]^2+\ldots\ldots\ldots$$

$$+1+\left[\frac{-1}{2}\right]\left[\frac{-c}{l}\right]+\frac{\left[\frac{-1}{2}\right]\left[\frac{-1}{2}-1\right]}{1.2}\left[\frac{-c}{l}\right]^2+\ldots\ldots\ldots$$

$$= 1 - \frac{c}{2l} + \frac{3c^2}{8l^2} + \ldots\ldots + 1 + \frac{c}{2l} + \frac{3c^2}{8l^2} + \ldots.$$

$= 2 + \frac{3c^2}{4l^2}$ approximately.

Question-35

Find the 5^{th} term in the expansion of $(1 - 2x^3)^{11/2}$.

Solution:

$$(1 - 2x^3)^{11/2} = 1 + \left[\frac{11}{2}\right](-2x^3) + \frac{\left[\frac{11}{2}\right]\left[\frac{2}{9}\right]}{1.2}(-2x^3)^2 + \frac{\left[\frac{11}{2}\right]\left[\frac{2}{9}\right]\left[\frac{7}{2}\right]}{1.2.3}(-2x^3)^3 + \frac{\left[\frac{11}{2}\right]\left[\frac{2}{9}\right]\left[\frac{7}{2}\right]\left[\frac{5}{2}\right]}{1.2.3.4}(-2x^3)^4$$

5^{th} term is $\frac{\left[\frac{11}{2}\right]\left[\frac{2}{9}\right]\left[\frac{7}{2}\right]\left[\frac{5}{2}\right]}{1.2.3.4}(-2x^3)^4 = \frac{1155}{8}x^{12}$

Question-36

Find the $(r + 1)^{th}$ term in the expansion of $(1 - x)^{-4}$.

Solution:

T_{r+1} in $(1 - x)^{-4}$

$(1-x)^{-4} = \frac{1}{6}[1.2.3 + 2.3.4.x + \ldots\ldots\ldots\ldots.(r + 1)(r + 2)(r + 3)x^r + \ldots\ldots\ldots]$

$\therefore T_{r+1} = \frac{(r+1)(r+2)(r+3)}{6}x^r$

Question-37

Show that $x^n = 1 + n\left[1-\frac{1}{x}\right] + \frac{n(n+1)}{1.2}\left[1-\frac{1}{x}\right]^2 + ...$

Solution:

R.H.S $= 1 + n\left[1-\frac{1}{x}\right] + \frac{n(n+1)}{1.2}\left[1-\frac{1}{x}\right]^2 + ...$

Put $y = 1-\frac{1}{x}$

$= 1 + ny + \frac{n(n+1)}{1.2}y^2 + ...$

$= (1 - y)^{-n}$

$= \left[1-\left[1-\frac{1}{x}\right]\right]^{-n}$

$= \left[\frac{1}{x}\right]^{-n}$

$= x^n$

$=$ L.H.S

Question-38

Find the sum to infinity of the series

(i) $1 + \frac{9}{8} + \frac{9.15}{8.16} + \frac{9.15.21}{8.16.24} + ...$

(ii) $1 - \frac{1}{5} + \frac{1.4}{5.10} - \frac{1.4.7}{5.10.15} + ...$

(iii) $\frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + ...$

Solution:

(i) Let $S = 1 + \frac{9}{8} + \frac{9.15}{8.16} + \frac{9.15.21}{8.16.24} + ...$

$= 1 + \frac{9}{6}\left(\frac{6}{8}\right) + \frac{\left(\frac{9}{6}\right)\left(\frac{5}{6}\right)}{1.2}\left(\frac{6}{8}\right)^2 + ...$

$= \left[1-\frac{6}{8}\right]^{-\frac{9}{6}}$

$\left[1-\frac{3}{4}\right]^{-\frac{3}{2}} = \left[\frac{1}{4}\right]^{-\frac{3}{2}} = 4^{\frac{3}{2}} = 4^1\ 4^{\frac{1}{2}} = \sqrt[4]{4} = 4(2) = 8$

(ii)Let S = $1 - \frac{1}{5} + \frac{1.4}{5.10} - \frac{1.4.7}{5.10.15} + ...$

$$= 1 - \left(\frac{1}{3}\right)\left(\frac{3}{5}\right) + \frac{\left[\frac{1}{3}\right]\left[\frac{4}{3}\right]}{1.2}\left[\frac{3}{5}\right]^2 + ...$$

$$= \left[1 + \frac{3}{5}\right]^{-\frac{1}{3}}$$

$$= \left[\frac{8}{5}\right]^{-\frac{1}{3}}$$

$$= \left[\frac{5}{8}\right]^{\frac{1}{3}}$$

(iii) Let S = $\frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + ...$

$$S + 1 = 1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + ...$$

$$= 1 + \left(\frac{3}{2}\right)\left(\frac{2}{4}\right) + \frac{\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{1.2}\left(\frac{2}{4}\right)^2 + ...$$

$$= \left(1 - \frac{2}{4}\right)^{-3/2} = \left(\frac{1}{2}\right)^{-3/2} = 2^{3/2}$$

$$S + 1 = 2^{3/2}$$

Therefore $S = 2^{3/2} - 1$

Question-39

(i) Show that the coefficient of x^n in the infinite series $1 + \frac{b+ax}{1!} + \frac{(b+ax)^2}{2!} + \frac{(b+ax)^3}{3!} + ...$ is $\frac{e^b a^n}{n!}$.

(ii) Show that $\left(1 + \frac{1}{2!} + \frac{1}{4!} + ...\right)^2 = 1 + \left(1 + \frac{1}{3!} + \frac{1}{5!} + ...\right)^2$.

(iii) Show that $2\left\{1 + \frac{(\log n)^2}{2!} + \frac{(\log n)^4}{4!} + ...\right\} = n + c$.

Solution:

(i) $1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + ... = e^y = e^{b+ax} = e^b . e^{ax} = e^b\left(1 + \frac{ax}{1!} + \frac{(ax)^2}{2!} + ...\right)$

Coefficient of $x^n = e^b . \left(\frac{a^2}{n!}\right)$

(ii) L.H.S = $\left(1 + \frac{1}{2!} + \frac{1}{4!} + ...\right)^2 = \left(\frac{e + e^{-1}}{2}\right)^2$

$$= 1 + \frac{e^2 + e^{-2} - 2}{4}$$

$$= \frac{4 + e^2 + e^{-2} - 2}{4}$$

$$= \frac{e^2 + e^{-2} + 2}{4}$$

= R.H.S

(iii) L.H.S $= 2\left\{1 + \frac{(\log n)^2}{2!} + \frac{(\log n)^4}{4!} + \ldots\right\}$

Put log n = y

$$2\left\{1 + \frac{(\log n)^2}{2!} + \frac{(\log n)^4}{4!} + \ldots\right\} = 2\left(\frac{e^y + e^{-y}}{2}\right)$$

$= e^y + e^{-y}$

$= e^{\log n} + e^{-\log n}$

$= e^{\log n} + e^{\log 1/n}$

$= n + \frac{1}{n}$

Question-40

Show that log a – log b = $\frac{a-b}{a} + \frac{1}{2}\left(\frac{a-b}{a}\right)^2 + \frac{1}{3}\left(\frac{a-b}{a}\right)^3 + \ldots\ldots\ldots$

Solution:

R.H.S = $\frac{a-b}{a} + \frac{1}{2}\left(\frac{a-b}{a}\right)^2 + \frac{1}{3}\left(\frac{a-b}{a}\right)^3 + \ldots\ldots\ldots$

Put y = $\frac{a-b}{a}$

$y + \frac{y^2}{2} + \frac{y^3}{3} + \ldots\ldots\ldots = -\log(1 - y)$

$= -\log\left(1 - \frac{a-b}{a}\right)$

$= -\log\left(\frac{b}{a}\right)$

$= \log\left(\frac{a}{b}\right)$

= log a – log b

= L.H.S

Question-41

Prove that $\log \frac{n+1}{n-1} = \frac{2n}{n^2+1} + \frac{1}{3}\left(\frac{2n}{n^2+1}\right)^3 + \ldots\ldots\ldots\ldots$

Solution:

R.H.S = $\frac{2n}{n^2+1} + \frac{1}{3}\left(\frac{2n}{n^2+1}\right)^3 + \ldots\ldots\ldots\ldots$

Put $\frac{2n}{n^2+1} = y$

$$y + \frac{y^3}{3} + \frac{y^5}{5} \ldots\ldots\ldots\ldots = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right)$$

$$= \frac{1}{2}\log\left(\frac{1+\frac{2n}{n^2+1}}{1-\frac{2n}{n^2+1}}\right)$$

$$= \frac{1}{2}\log\left(\frac{n^2+1+2n}{n^2+1-2n}\right)$$

$$= \frac{1}{2}\log\left(\frac{n+1}{n-1}\right)^2$$

$$= \log\left(\frac{n+1}{n-1}\right)$$

= L.H.S

Question-42

Find the sum to infinity the series $\frac{1}{x-1} + \frac{1}{3}\frac{1}{[x-1]^3} + \frac{1}{5}\frac{1}{[x-1]^5} + \ldots\ldots\ldots\ldots\ldots\ldots$

Solution:

$\frac{1}{x-1} + \frac{1}{3}\frac{1}{(x-1)^3} + \frac{1}{5}\frac{1}{(x-1)^5} + \ldots\ldots\ldots\ldots\ldots\ldots$

Put $y = \frac{1}{x-1}$

$$\frac{1}{y} + \frac{1}{3}\frac{1}{y^3} + \frac{1}{5}\frac{1}{y^5} + \ldots\ldots\ldots\ldots = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right)$$

$$= \frac{1}{2}\log\left(\frac{1+\frac{1}{1-x}}{1-\frac{1}{1-x}}\right)$$

$$= \frac{1}{2}\log\left(\frac{1-x+1}{1-x-1}\right)$$

$$= \frac{1}{2}\log\left(\frac{2-x}{-x}\right)$$

$$= \frac{1}{2}\log\left(\frac{x+2}{x}\right)$$